Swarm approximation for weakly ionized plasmas

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Upper bounds on errors in the moments of the electron velocity distribution function, coming from using the conventional two term approximation scheme to solve the swarm kinetic equation, are found in terms of the mass ratio of an electron and a neutral particle. The results significantly improve existing estimates, and show that the conventional approximation is highly accurate for elastic collisions between electrons and neutrals. The applicability of the collision integral in a differential form is discussed. [S1063-651X(98)11907-4]

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I. INTRODUCTION

The study of the electron distribution function (EDF) in weakly ionized plasmas subjected to external fields was started at the beginning of this century by Pidduck [1], after Townsend experimentally found an electron mean energy greatly exceeding the energy of neutral gas particles. Assuming a very low electron density, and considering only elastic collisions of the electron with neutral particles, this problem is usually reduced to solving the Boltzmann equation with the collision integral replaced by a differential operator due to the great disparity in the masses of the colliding particles. Keeping only the two lowest order terms in the Legendre series expansion of the EDF made it possible for Druyvesteyn in 1930 to obtain the distribution [2] for velocity independent collision cross sections at zero ambient temperature. This was generalized later [3] by Davydov. A detailed description of this technique can be found in Ref. [4]. The simplicity and effectiveness of this method made the ''two-term'' approximation very popular, and researchers in numerous publications have reported about its successful applications with constant and time varying electric and magnetic fields, and even for treating ''slightly'' inelastic collisions when an electron loses a relatively small fraction of its energy in collisions with molecules having rotational and vibrational degrees of freedom.

The accuracy of this two-term calculation scheme is clearly very important. All those using it have commented on this to some degree. In our recent work [5], we found that the electron mobility and stationary energy given by the two-term approximation cannot differ by more than $\pm 12\%$ and $\pm 18\%$, respectively, from their exact values when the background temperature can be neglected and the collision cross section is energy independent. These rather rough bounds are valid for all mass ratios less than about $\frac{1}{10}$. However, it was shown via Monte Carlo simulation [6] for a helium plasma that the accuracy is significantly better. Here we rigorously prove that the error should scale like the mass ratio of an electron to a neutral particle and calculate the prefactors explicitly.

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II. PROPERTIES OF THE REDUCED BOLTZMANN EQUATION

Let us consider a weakly ionized plasma subjected to a spatially homogeneous electric field of intensity E. The one species neutrals, whose number density is N and the mass of a particle of which is M, are assumed to have a Maxwellian velocity distribution with a temperature T; the electron collisions with neutrals are spherically symmetric and elastic with a total cross section $\sigma(v)$. If m is the mass of an electron, a straightforward expansion [7] of the Boltzmann collision integral in terms of the small parameter $\epsilon = m/M$, neglecting all terms of order ϵ^2, yields the following kinetic equation for the stationary EDF $f(x,v)$

$$-\frac{eE}{mN}\left(x\,\frac{\partial f}{\partial v}+\frac{1-x^2}{v}\,\frac{\partial f}{\partial x}\right) = \frac{1}{v^2}\,\frac{d}{dv}\left[\epsilon\sigma(v)v^4\left(f_0+\frac{kT}{mv}\,\frac{df_0}{dv}\right)\right]+v\,\sigma(v)(f_0-f\ ). \tag{1}$$

Here x is the cosine of the angle between the direction of the electron velocity $\mathbf{v}$ and the field direction, and $f_0(v)$ is the spherically symmetric part of $f(x,v)$. Sometimes [8] ϵ is treated as a velocity dependent effective parameter representing the energy transfer in almost elastic collisions.

The formal expansion of the collision integral in ϵ is dubious mathematically, but a kinetic equation very similar to Eq. (1) can be obtained in different ways. The derivation in Ref. [9] is based on the two-term Legendre expansion of the EDF and the Fokker-Planck approximation. The differential operator on the right side of Eq. (1) is applied not to f_0 but to $f(x,v)$, though this does not make large difference because the higher harmonics of $f(x,v)$ are also of a higher order in ϵ. We shall work with Eq. (1), and compare its solution with the conventional one, assuming the existence of a solution $f(x,v)$ of Eq. (1) which decays rapidly as $v\to\infty$ and closely describes the physical EDF. The collision parameters here are set to be velocity independent, $\sigma=$const and $\epsilon=$const. Our results will imply that one cannot hope to noticeably improve the solution of Eq. (1) by replacing the conventional approach by any more advanced technique when the parameter of inelasticity (ϵ here) is small enough.

From now on we use dimensionless quantities

$$y=v\epsilon^{1/4}\left(\frac{mN\sigma}{eE}\right)^{1/2},\quad t=\frac{kTN\sigma\sqrt{\epsilon}}{eE}.$$

Equation (1) for $f(x,y)$ now can be written in the form

$$-\sqrt{\epsilon}\left(x\frac{\partial f}{\partial y}+\frac{1-x^2}{y}\frac{\partial f}{\partial x}\right)=\epsilon\frac{1}{y^2}\frac{d}{dy}\left[y^4\left(f_0+\frac{t}{y}\frac{df_0}{dy}\right)\right]+y(f_0-f). \quad (2)$$

The Legendre series expansion

$$f(x,y)=\sum_{l=0}^{\infty}f_l(y)P_l(x), \quad (3)$$

where

$$f_l(y)=\frac{2l+1}{2}\int_{-1}^{1}f(x,y)P_l(x)dx,\quad \text{for } l=0,1,2,\ldots, \quad (4)$$

converts Eq. (2) into the set of coupled equations for the Legendre harmonics

$$\sqrt{\epsilon}\left[\frac{l}{2l-1}\left(\frac{df_{l-1}}{dy}-\frac{l-1}{y}f_{l-1}\right)+\frac{l+1}{2l+3}\left(\frac{df_{l+1}}{dy}+\frac{l+2}{y}f_{l+1}\right)\right]=y\,f_l(y),$$
$$l=1,2,\ldots\,. \quad (5)$$

In addition (for $l=0$) there is one more equation

$$\frac{df_1}{dy}+\frac{2}{y}f_1=-3\sqrt{\epsilon}\frac{1}{y^2}\frac{d}{dy}\left[y^4\left(f_0+\frac{t}{y}\frac{df_0}{dy}\right)\right],$$

whose solution represents $f_1(y)$ exclusively in terms of $f_0(y)$:

$$f_1(y)=-3\sqrt{\epsilon}y^2\left(f_0+\frac{t}{y}\frac{df_0}{dy}\right). \quad (6)$$

In Eq. (6), we used the decay of $f(x,v)$ at infinity to eliminate the constant of integration.

The original Boltzmann equation has only a non-negative solution, and if Eq. (1) is a good version of the kinetic equation then $f(x,y)\geqslant 0$ everywhere. In a weaker sense, one might allow a ''nonphysical'' behavior of $f(x,y)$ somewhere in its domains $-1\leqslant x\leqslant 1$, and $0\leqslant y<\infty$ if this does not noticeably affect meaningful integrals of $f(x,y)$. The long history of using Eq. (1) successfully suggests that, at least for small ϵ, this is correct. Nevertheless it is easy to show that the time dependent analog of Eq. (1) with $\sigma\not\equiv$const does not conserve positivity for special initial conditions. Therefore, one should be careful using Eq. (1), especially when the parameter of inelasticity ϵ of the electron collision is not very small.

Let us explore here the situation when $f(x,y)\geqslant 0$, as we did in Ref. [5]. It is obvious that

$$|f^l(y)|\leqslant(2l+1)f_0(y), \quad (7)$$

because $|P_l(x)|\leqslant 1$. By virtue of Eqs. (6) and (7), we have

$$f_0(y)\geqslant\sqrt{\epsilon}y^2\left|f_0(y)+\frac{t}{y}\frac{df_0}{dy}\right|. \quad (8)$$

This inequality implies that when $y>\epsilon^{-1/4}$, the tail of the EDF is a Maxwellian with the temperature of neutrals. When $T=0$ (this is quite common in applications because the electrons easily gain a much higher energy than the background gas), $f_0(y)$ is a function with compact support like $f(x,y)$ and all harmonics $f_l(y)$. Let us show that the boundary of the region where $f(x,y)\not\equiv 0$ is $y=$const. Assuming the opposite, $y=Y(x)\neq$const, we chose a point (x,y) in the domain of $f(x,y)$ such that $Y(x)<y\leqslant\text{Max}[Y(x)]$. In the vicinity of this point Eq. (1) is reduced to

$$\frac{\epsilon}{y^2}\frac{d}{dy}(y^4f_0)+yf_0=0. \quad (9)$$

The solution of Eq. (9), $f_0(y)=Cy^{-4-1/\epsilon}$, is not equal to zero at any finite $y=Y(x)$, therefore, $Y=$const. This asymptotics of EDF, obtained as the solution of Eq. (1) might not be practically important for purely elastic collisions when $\epsilon\leqslant 10^{-4}$. However if one wants to use Eq. (1) for $\epsilon\sim 0.03$, and the plasma pressure is about 0.1 Torr, the boundary $y\geqslant\epsilon^{-1/4}$ means that all electrons with an energy greater than 1 eV will be in the nonphysical region. In this work we do not assume $f(x,y)\geqslant 0$ for all x,y, or assume compact support for $f(x,y)$.

Set (5) allows the expansion of $f_l(y)$ in Taylor's series

$$f_l(y)=\sum_{i=0}^{\infty}a_i^l y^{2i},$$

where the coefficients a_i^l can be expressed through a_i^0:

$$\sqrt{\epsilon}\left[\frac{l(2i+1-l)}{2l-1}a_i^{l-1}+\frac{(l+1)(2i+2+l)}{2l+3}a_i^{l+1}\right]=a_{i-1}^l,\quad l,i=1,2,\ldots \quad (10)$$

and

$$\frac{l(1-l)}{2l-1}a_0^{l-1}+\frac{(l+1)(1+2)}{2l+3}a_0^{l+1}=0. \quad (11)$$

Equation (6) implies

$$a_0^1=0,\quad a_i^1=-3\sqrt{\epsilon}a_{i-1}^0,\quad i\geqslant 1$$

and therefore one can easily see from Eq. (11) that, for all $l\geqslant 1$,

$$a_0^l=0. \quad (12)$$

All higher harmonics vanish at $y=0$, except $a_0^0=f_0(0)=f(x,0)$.

In the conventional approximation scheme, one neglects all the higher harmonics $f_l(y)$ starting from $l=2$. This reduces the set of equations (5) and (6) to the equation

$$(1+3y^2t)\,\frac{df_0}{dy}+3y^3f_0=0,$$

whose solution for the particular case of zero background temperature $t=0$ is the well known Druyvesteyn's distribution

$$D(y)=\exp\left(-\frac{3}{4}\,y^4\right). \tag{13}$$

We shall show here how good approximation (13) is for a computation of the main moments of the EDF. The main macroscopic plasma parameters are the electron drift

$$w=\left(\frac{eE}{mN\sigma}\right)^{1/2}\epsilon^{1/4}\int_0^\infty y^5\left(f_0+\frac{t}{y}\,\frac{df_0}{dy}\right)dy \tag{14}$$

and the mean energy

$$W=\frac{eE}{2N\sigma\sqrt{\epsilon}}\int_0^\infty y^4\;f_0(y)dy, \tag{15}$$

where we set the normalization

$$\int_0^\infty y^2\;f_0(y)dy=1. \tag{16}$$

All these quantities can be evaluated through the first several moments of $f_0(y)$.

III. MOMENTS OF THE HARMONICS

Using Eq. (6) with $t=0$, Eq. (5) for $l=1$ has the form

$$\frac{df_0}{dy}+3y^3f_0=-\frac{2}{5}\left(\frac{df_2}{dy}+\frac{3}{y}\,f_2\right). \tag{17}$$

The formal solution of Eq. (17) is

$$f_0(y)=AD(y)-\frac{2}{5}\,f_2(y)+\frac{6}{5}\int_c^y\left(s^3-\frac{1}{s}\right)\frac{D(y)}{D(s)}\,f_2(s)ds \tag{18}$$

where $A=[f_0(c)+\frac{2}{5}f_2(c)]/D(c)$, and parameter $c>0$ is arbitrary.

We want to evaluate the moments

$$M_l(q)=\int_0^\infty y^q f_1(y)dy,\quad M_0(q)\equiv M(q),\quad M(2)=1.$$

One needs [see Eqs. (14) and (15)] only $M(3)$, $M(4)$ and $M(5)$ for the computation of w and W. Equation (17) yields

$$qM(q-1)-3M(q+3)=2\,\frac{3-q}{5}\,M_2(q-1),\quad q>0. \tag{19}$$

In Ref. [5] we derived more general relations than Eq. (19) for $t\neq 0$, where we assumed $f_0(y)\geqslant 0$. Manipulating them and using the convexity of ln $M(q)$ for the interpolations, we obtained the two-sided bounds on $M(3)$, $M(4)$, and $M(5)$, and therefore for w and W, with an accuracy of about 10–20 %. We shall obtain much better bounds here for small enough ϵ.

Setting the notation

$$U(q)=\int_0^\infty y^qD(y)dy=\frac{1}{4}\left(\frac{4}{3}\right)^{(q+1)/4}\Gamma\left(\frac{q+1}{4}\right),$$

we now introduce the Druyvesteyn distribution with the same normalization (16) as $f_0(y)$,

$$f_D(y)=\frac{1}{U(2)}\,D(y),$$

and its moments

$$M_D(q)=\frac{1}{U(2)}\,U(q). \tag{20}$$

Our goal is to show that for $q=2$, 3, 4, and 5, the differences between the moments $M(q)$ of the (unknown) solution $f_0(y)$ and their Druyvesteynian counterparts, $M_D(q)$, scale as ϵ

$$|M(q)-M_D(q)|\leqslant p(q,\epsilon)\epsilon, \tag{21}$$

and calculate upper bounds for prefactors $p(q,\epsilon)$. Unfortunately this requires quite a laborious technique which involves not only all moments $M_l(q)$, but also the moments of products $f_l(y)f_n(y)$ and absolute values $|f_l(y)|$.

We may evaluate the moments by the direct integration of Eq. (18). Taking $c=1$ and using the estimation of incomplete gamma functions [10], we obtain, for $-1<q\leqslant 7$, the inequality

$$|M(q)-AU(q)|\leqslant\tfrac{4}{5}\bar{M}_2(q)+C(q)\bar{M}_2(3), \tag{22}$$

where

$$\bar{M}_2(q)=\int_0^\infty y^q|f_2(y)|dy,$$

$$C(q)=\mathrm{Max}\left\{0,\,2\,\frac{q-3}{15}\left(\frac{4}{3}\right)^{(q-7)/4}\Gamma\left(\frac{q-3}{4}\right)\right\}.$$

Taking $q=2$ in Eq. (22), we find bounds for A:

$$1-\tfrac{4}{5}\bar{M}_2(2)\leqslant AU(2)\leqslant 1+\tfrac{4}{5}\bar{M}_2(2). \tag{23}$$

With the help of Eq. (23), inequality (22) for all q can be replaced by a stronger one:

$$|M(q)-M_D(q)|\leqslant\frac{4}{5}\,\bar{M}_2(q)+C(q)\bar{M}_2(3)+\frac{4U(q)}{5U(2)}\,\bar{M}_2(2). \tag{24}$$

Inequality (24) shows that estimate (21) can be obtained if the integrals $M_2(q)$ are of order of ϵ comparing with $M(q)$, or even generally if f_2 is of order ϵf_0. The computation of the upper bounds on $M_2(q)$ is sketched in the Appendix.

IV. BOUNDS FOR THE MOMENTS

We can now solve the problem of bounding the moments of the EDF around the moments M_D corresponding the normalized Druyvesteynian. This requires relation (24), where the left side terms can be bounded from above, while for the Dryuvesteyn distribution we have, in virtue of Eq. (20)

$$M_D(3)\approx 0.8769, \quad M_D(4)\approx 0.8541, \quad M_D(5)\approx 0.8974. \tag{25}$$

Using Eq. (A1) with $p=1$, Eqs. (A11)–(A13) and (A32), we easily find

$$\begin{aligned}\bar{M}_2(2) &\leqslant \left(\frac{2\pi}{3\sqrt{3}}\,[M_{22}(3)+M_{22}(6)]\right)^{1/2}\\ &\leqslant \left(\frac{2\pi}{3\sqrt{3}}\,[M_{22}(3)+M_{22}^{1/4}(3)M_{22}^{3/4}(7)]\right)^{1/2}\\ &\leqslant \epsilon\left(\frac{2\pi}{3\sqrt{3}}\,M_{00}(3)[B(3)+B^{1/4}(3)B^{3/4}(7)]\right)^{1/2}\\ &\leqslant \frac{10.8\epsilon}{1-20\epsilon},\end{aligned} \tag{26}$$

and, in the same way,

$$\begin{aligned}\bar{M}_2(4) &\leqslant \epsilon\left(\frac{\pi}{7\sin(\pi/7)}\,M_{00}(3)[B(3)+B^{1/4}(7)B^{3/4}(11)]\right)^{1/2}\\ &\leqslant \frac{16.2\epsilon}{1-20\epsilon},\end{aligned} \tag{27}$$

$$\begin{aligned}\bar{M}_2(5) &\leqslant \epsilon\left(\frac{\pi/9}{\sin(\pi/9)}\,M_{00}(3)[B(3)+B^{1/2}(11)B^{1/2}(13)]\right)^{1/2}\\ &\leqslant \frac{24.3\epsilon}{1-20\epsilon}.\end{aligned} \tag{28}$$

Substituting Eq. (A32) into Eq. (A31) gives

$$\bar{M}_2(3)\leqslant \frac{12.5\epsilon}{1-20\epsilon}. \tag{29}$$

With the help of inequalities (A32), (26)–(29), and (24) we obtain finally the prefactors in Eq. (21),

$$p(3,\epsilon)=\frac{17.6}{1-20\epsilon}, \quad p(4,\epsilon)=\frac{25.3}{1-20\epsilon}, \quad p(5,\epsilon)=\frac{32.3}{1-20\epsilon}, \tag{30}$$

uniformly for $\epsilon\leqslant 0.02$.

V. DISCUSSION

We compare the bounds (21) and (30) with the results of Ref. [5] for the case corresponding to our present calculation. By virtue of Eqs. (14) and (15), the drift velocity and mean energy of electrons when $t=0$ (negligible background temperature T) are, respectively,

$$w=\epsilon^{1/4}\left(\frac{eE}{mN\sigma}\right)^{1/2}M(5), \quad W=\frac{eE}{2N\sigma\sqrt{\epsilon}}\,M(4).$$

Using Eqs. (21), (30), and (25), we can write these quantities in terms of their Druyvesteyn's approximations

$$\left|\frac{w}{w_D}-1\right|\leqslant\frac{36\epsilon}{1-20\epsilon}, \quad \left|\frac{W}{W_D}-1\right|\leqslant\frac{30\epsilon}{1-20\epsilon}. \tag{31}$$

Bounds (31) are valid when $\epsilon<0.02$ and also $T=0$, i.e., the strong field regime $kT\ll\epsilon^{-1/2}eE/N\sigma$. For the same conditions the results of Ref. [5] give

$$-(0.063+0.5\epsilon)\leqslant\frac{w}{w_D}-1\leqslant 0.114,$$

$$-(0.172+\epsilon)\leqslant\frac{W}{W_D}-1\leqslant 0.171. \tag{32}$$

For small ϵ, which is valid for purely elastic processes, the present bounds not only show convergence of the Druyvesteyn approximation scheme, but also become very precise. The errors according to Eq. (31) do not exceed 3.7% for $\epsilon=0.001$, or 0.9% for ϵ corresponding to a helium plasma and only a small fraction of one percent in heavier gases; however, our results can be applied for the real gases only qualitatively. In a more general setting, treating ϵ as a parameter of energy loss in slightly inelastic collisions we can also consider a situation when $\epsilon\sim 0.003$–0.1. In this case one can use Eq. (32), where the estimates match those of Eq. (31) at the low end of the range. Bounds (32) have an advantage of being almost independent of ϵ, and they are even better than Eq. (31) when $\epsilon>0.003$. However, one should be very cautious in using the kinetic equation in its differential form for such values of ϵ.

Our rigorous results are obtained for the spatially homogeneous stationary plasma at a zero (very low, $kT\ll eE/N\sigma\sqrt{\epsilon}$) temperature of the background gas, and a constant cross section of the electron-neutral particle collisions. We use the method of moments, whose generalization to velocity dependent collisions is straightforward when the cross section can be described by a power law

$$\sigma(v)=\sigma_0\left(\frac{v}{v_0}\right)^r, \quad r\geqslant -1, \tag{33}$$

or even a linear combination of functions (33), which gives a freedom in modeling $\sigma(v)$. The chain of equations for the harmonics of an EDF similar to Eq. (5) was derived in Ref. [5], as well as expressions for w_D and W_D. We do not see any serious obstacles to expanding our method in this case, and our preliminary study shows that the errors in using the two-term approximation for Eq. (33), $|w/w_D-1|$ and $|W/W_D-1|$, scale as ϵ here too. While we do not know how to treat a more general $\sigma(v)$, the present result and the promising perspective for model (33) give us confidence in the conventional scheme of computation for small ϵ.

The case $T\neq 0$ and σ in form (33) can be studied by the method of moments (see Ref. [5]). It seems that one can obtain similar estimates straightforwardly, though the calculation of prefactors might be more difficult. Regarding nonstationary and spatially inhomogeneous problems, we note that if the characteristic time and length intervals of inhomogeneity are significantly larger than

$$\frac{\epsilon^{-1/2}}{N\sigma}, \quad \text{and} \quad \epsilon^{-3/4}\left(\frac{m}{eEN\sigma}\right)^{1/2},$$

respectively, then the swarm behaves locally as it were stationary and homogeneous, so the present results can be applied locally. When these requirements are not satisfied, the electron distribution function will have a more complicated structure, and should be studied by different methods.

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APPENDIX

1. Binary moments

We cannot estimate $\bar{M}_2(q)$ in Eq. (24) directly; instead, using the Helder inequality

$$\int y^q |f_l(y)| dy \leqslant \left(\int \frac{y^p dy}{1+y^{p+2}} \int y^{2q-p}(1+y^{p+2}) f_l^2(y) dy \right)^{1/2},$$

we express the upper bound on $\bar{M}_2(q)$,

$$\bar{M}_l(q) \leqslant \left(\frac{\pi/(p+2)}{\sin[\pi/(p+2)]} [M_{ll}(2q-p)+M_{ll}(2q+2)] \right)^{1/2}, \quad p>-1, \tag{A1}$$

through the binary moments

$$M_{kl}(q) = \int_0^\infty y^q f_k(y) f_l(y) dy. \tag{A2}$$

Together with $M_{kl}(q)$, we will use strictly positive moments $\underline{M}_{kl}(q)$, which are computed with the absolute values of $f_k(y)$ and $f_l(y)$ in Eq. (A2).

Let us multiply each equation (5) by a corresponding term $f_l(y) y^{q-1}/(2l+1)$, integrate over y, and find the infinite sums from $l=2$ to ∞. The result,

$$\frac{1}{5}\left(1-2\epsilon\frac{5-q}{5}\right)\int_0^\infty y^q f_2^2 dy + \sum_{l=3}^\infty \frac{1}{2l+1}\int_0^\infty y^q f_l^2 dy = \frac{2\epsilon}{5}\int_0^\infty (3y^4-1) y^q f_0 f_2 dy + \sqrt{\epsilon}(3-q) \times \sum_{l=2}^\infty \frac{l+1}{(2l+1)(2l+3)} \int_0^\infty y^{q-2} f_l f_{l+1} dy, \tag{A3}$$

is valid for $q \geqslant -2$. In particular, when $q=3$, Eq. (A3) reduces to

$$\sum_{l=2}^\infty \frac{1}{2l+1} M_{ll}(3) < \frac{2\epsilon}{5-4\epsilon}[3M_{02}(7)-M_{02}(4)]. \tag{A4}$$

Keeping in mind the evaluation of $M_{22}(q)$, we shall find upper bounds for the set of series with non-negative terms,

$$P(q) \equiv M_{22}(q) + \frac{5}{2}\sum_{l=3}^\infty \frac{1}{2l+1} M_{ll}(q),$$

starting from $q=3$ and adding 4 to each previous q successively. The initial term with $q=3$ can be obtained from Eq. (A4):

$$P(3) \leqslant 2P(3) - M_{22}(3) \leqslant \frac{2\epsilon}{1-4\epsilon/5}[3M_{02}(7)-M_{02}(3)]. \tag{A5}$$

Using, in Eq. (A3), Cauchy's inequality in the form

$$\frac{\sqrt{\epsilon}(l+1)}{(2l+1)(2l+3)} y^{q-2} f_l f_{l+1} \leqslant \frac{3}{2\sqrt{35}}\left(r\frac{\epsilon y^{q-4} f_l^2}{2l+1} + \frac{1}{r}\frac{y^q f_{l+1}^2}{2l+3}\right), \quad l \geqslant 2, \; r>0,$$

and taking $r=3(q-3)/\sqrt{35}$, we obtain for $q \geqslant 5$ the recursive inequality

$$P(q) \leqslant 2\epsilon[3M_{02}(q+4)-M_{02}(q)] + \frac{9\epsilon(q-3)^2}{35} P(q-4). \tag{A6}$$

By analogy with Eq. (19) one has an exact relation

$$3M_{00}(q+4) - \frac{q+1}{2} M_{00}(q) = 2\frac{q-2}{5} M_{02}(q) - \frac{6}{5} M_{02}(q+4) + 2\frac{q-5}{25} M_{22}(q), \tag{A7}$$

which can be obtained by simple manipulations with Eq. (17). Equation (A7) together with inequalities (A5) and (A6) will be used systematically for finding bounds on moments M_{22} in terms of M_{00} in the form

$$M_{22}(q) < B(q)\epsilon^2 M_{00}(3), \quad B \geqslant 0. \tag{A8}$$

In bounds (A8) we need obviously only a rough estimate for $M_{00}(q)$, but it must be expressed in terms of a known function, $D(y)$ in our case.

We multiply Eq. (18) (with $c=1$) by $f_2(y)$, and integrate with the factor y^q $(q>-1)$:

$$M_{00}(q)=A\int_0^\infty y^q D(y)f_0(y)dy-\frac{2}{5}M_{02}(q)+\frac{6}{5}\int_0^\infty f_0(y)y^q dy\int_1^y\left(s^3-\frac{1}{s}\right)\frac{D(y)}{D(s)}f_2(s)ds. \tag{A9}$$

In the last term of Eq. (A9), by splitting the interval of the first integration in $y=1$, changing the order of integration, and applying Helder's inequalities, the upper bound for the integral can be estimated as

$$\left(\frac{M_{00}(3)}{2q-2}\right)^{1/2}\int_0^1(1-s^4)s^{q-2}e^{3s^4/4}|f_2(s)|ds+\left(\frac{M_{00}(2q-3)}{6}\right)^{1/2}\int_1^\infty(s^3-s^{-1})|f_2(s)|ds.$$

Noticing that $(1-s^4)\exp(3s^4/4)\leqslant 1$, we obtain

$$M_{00}(q)\leqslant A\sqrt{V(q)M_{00}(q)}+\frac{2}{5}|M_{02}(q)|+\frac{6}{5}\left[\left(\frac{M_{00}(3)}{2q-2}\right)^{1/2}\int_0^1 s^{q-2}|f_2(s)|ds+\left(\frac{M_{00}(2q-3)}{6}\right)^{1/2}\int_1^\infty s^3|f_2(s)|ds.\right], \tag{A10}$$

where

$$V(q)=\int_0^\infty y^q D^2(y)dy=\frac{1}{4}\left(\frac{2}{3}\right)^{(q+1)/4}\Gamma\left(\frac{q+1}{4}\right),$$

and we used the convexity of $\ln[\overline{M}_{lk}(q)]$, implying the relations

$$\overline{M}_{lk}(q)\leqslant\sqrt{M_{ll}(p)M_{kk}(r)},\quad p+r=2q,\ \ p,r\geqslant 0, \tag{A11}$$

$$M_{ll}^{a+b}(q)\leqslant M_{ll}^b(q-a)M_{ll}^a(q+b),\quad q\geqslant 0,\ \ a,b\geqslant 0. \tag{A12}$$

We apply inequalities (A11) and (A12) for the interpolation because the estimations of moments $M_{lk}(q)$ will be done for several discrete values of q.

Let us outline the following steps: in Appendix A 2 quite a tricky manipulation with relations (A5)–(A7) allows us to find the upper bounds on $M_{22}(q)$ for $q=3$, 7, 11, and 13 in terms of $M_{00}(3)$. Using these bounds and Eq. (A10) in Appendix A 3, we evaluate the upper bound for $M_{00}(3)$ through the Druyvesteyn moments (20), i.e., in dimensionless numbers.

2. Estimates for $M_{22}(q)$

Here we calculate $B(q)$ for Eq. (A8), and afterwards express $M_{00}(3)$ through U and V with the help of Eq. (A10). Inequalities (A5) and (A6) together with Eqs. (A11) and (A12) allow us to find bounds on $M_{22}(q)$ in terms of $M_{00}(q)$, $M_{00}(q+4)$, and $M_{00}(q+8)$. The crucial point here is to reach the closure and evaluate $M_{00}(q+4)$ through $M_{00}(q)$. The procedure will require several steps of recursion, and require using the positivity of all $M_{ll}(q)$ in inequalities (A5) and (A7). For $q=3$ this is not difficult: by substituting

$$3M_{02}(7)-M_{02}(3)=\tfrac{5}{2}[2M_{00}(3)-3M_{00}(7)]-\tfrac{2}{5}M_{22}(3)>0 \tag{A13}$$

into Eq. (A5), one obtains

$$M_{00}(7)<\frac{2}{3}M_{00}(3)-\frac{P(3)}{15\epsilon}. \tag{A14}$$

For $q=7$, combining Eqs. (A5) and (A6), we can write

$$P(7)\leqslant 8\epsilon M_{02}(7)+\frac{4\epsilon}{5}M_{22}(7)+5\epsilon[4M_{00}(7)-3M_{00}(11)]+\frac{144\epsilon}{35}P(3). \tag{A15}$$

Using Cauchy's inequality

$$8\epsilon f_0 f_2\leqslant\frac{1}{1-4\epsilon/5}(4\epsilon f_0)^2+(1-4\epsilon/5)f_2^2$$

for the estimation of $M_{02}(7)$:

$$8\epsilon M_{02}(7)\leqslant\frac{16\epsilon^2}{1-4\epsilon/5}M_{00}(7)+\left(1-\frac{4\epsilon}{5}\right)M_{22}(7),$$

from Eq. (A15) we derive

$$M_{00}(11)\leqslant\frac{4}{3(1-4\epsilon/5)}M_{00}(7)+\frac{48}{175}P(3). \tag{A16}$$

Now, using Eq. (A11) in Eq. (A5), we have

$$P(3)\leqslant\frac{2\epsilon}{1-4\epsilon/5}\times\sqrt{[9M_{00}(11)-6M_{00}(7)+M_{00}(3)]P(3)}. \tag{A17}$$

The solution of two inequalities (A16) and (A17) with the help of Eq. (A14) yields the upper bounds

$$M_{22}(3)\leqslant P(3)\leqslant 20\left(1+\frac{3}{2}\epsilon\right)\epsilon^2 M_{00}(3), \tag{A18}$$

$$M_{00}(11)<\frac{4}{3}M_{00}(7)+\frac{32}{45}\epsilon(1+9\epsilon)M_{00}(3),$$

or

$$M_{00}(11)<\frac{8M_{00}(3)}{9(1-4\epsilon/5)}, \tag{A19}$$

where here and everywhere further we assume $\epsilon \leqslant 0.02$ for a simplification of the estimates. Relations (A15), (A17), and (A18), in addition to Eq. (A14), allow us to obtain the lower bound on $M_{00}(7)$,

$$\frac{2}{3}[1-2\epsilon(1+3.4\epsilon)] \leqslant \frac{M_{00}(7)}{M_{00}(3)} \leqslant \frac{2}{3}. \tag{A20}$$

This completes the case $q=3$. In order to have an upper bound (A8) for $M_{22}(7)$, and upper bound on $M_{00}(15)$, we need to go to $q=11$ in Eqs. (A6) and (A7):

$$P(11) \leqslant 2\epsilon[3M_{02}(15)-M_{02}(11)] + \frac{576\epsilon}{35} P(7),$$

$$M_{00}(15)-2M_{00}(11) = \frac{6}{5} M_{02}(11) - \frac{2}{5} M_{02}(15) + \frac{4}{25} M_{22}(11).$$

A technique similar to previous one yields first

$$M_{00}(15) < 2\left[1+\frac{32\epsilon}{15(1-12\epsilon/5)}\right] M_{00}(11) + \frac{192}{175} P(7), \tag{A21}$$

and after returning to $q=7$, the bound on $P(7)$,

$$P(7) \leqslant \frac{136}{3}\epsilon^2 \frac{1+7\epsilon}{1-40\epsilon^2} M_{00}(3). \tag{A22}$$

The same method for $q=15$ in Eqs. (A6) and (A7), returning to $P(11)$, leads, after quite trivial but intricate computations, to inequalities

$$M_{00}(19) \leqslant \frac{8}{3}\frac{1-2\epsilon/5}{1-4\epsilon} M_{00}(15) + \frac{432}{175} P(11) \tag{A23}$$

and

$$P(11) \leqslant \frac{1184}{9}\epsilon^2 \frac{1+22\epsilon}{1-89\epsilon^2} M_{00}(3). \tag{A24}$$

We need moments from $M(2)$ to $M(5)$, and therefore, according to Eqs. (24) and (A1), upper bounds on $M_{22}(q)$ up to $q=12$. This permits us to make a shorter step toward $P(13)$, again applying the same technique: begin by jumping in Eq. (A6) from $P(13)$ to $P(17)$, and use Eq. (A7) and Cauchy's inequality to calculate the bound

$$M_{00}(21) \leqslant \left[3+\frac{196\epsilon}{3(5-24\epsilon)}\right] M_{00}(17) + \frac{84}{25} P(13);$$

then return to $q=13$ in Eq. (A6) and there use Eqs. (A11) and (A12) to obtain, eventually

$$P(13) < 249\epsilon^2 \frac{1+70\epsilon}{1-121\epsilon^2}. \tag{A25}$$

The main results of this section are

$$B(3) = 20(1+1.5\epsilon), \quad B(7) = \frac{136}{3}\frac{1+7\epsilon}{1-40\epsilon^2},$$

$$B(11) = \frac{1184}{9}\frac{1+22\epsilon}{1-89\epsilon^2}, \quad B(13) = 249\frac{1+70\epsilon}{1-121\epsilon^2} \tag{A26}$$

uniformly for all $\epsilon \leqslant 0.02$.

3. Evaluation of $M_{00}(3)$

Equations (A26) together with inequalities (24), (A1), (A11), and (A12), open the possibility to estimate the deviations of moments $M(3)$, $M(4)$, and $M(5)$ from their Druyvesteyn approximations in terms of $\epsilon M_{00}(3)$. We cannot express $M_{00}(3)$ through A and $\underline{V}(3)$ directly from Eq. (A10) because for $q=3$ the term $M_2(1)$ emerges in Eq. (A10) whose estimation with the help of Eq. (A1) requires $M_{22}(2)$. Instead, using Eqs. (A10) and (A20), we estimate the moment $M_{00}(5)$,

$$M_{00}(5) \leqslant A\sqrt{V(5)M_{00}(5)} + \frac{2}{5}|M_{02}(5)| + \frac{3}{5\sqrt{2}}\overline{M}_2(3)\sqrt{M_{00}(3)}, \tag{A27}$$

and find the upper bound on $M_{00}(3)$ in terms of $M_{00}(5)$. One needs inequalities (A12), (A19), and (A20) for this:

$$M_{00}(7) \leqslant M_{00}^{2/3}(5) M_{00}^{1/3}(11),$$

and therefore

$$M_{00}(5) \geqslant \left(\frac{1-4\epsilon/5}{3}\right)^{1/2} [1-2\epsilon(1+3.4\epsilon)]^{3/2} M_{00}(3). \tag{A28}$$

Now we express $M_{02}(5)$, $\overline{M}_2(3)$, and A in Eq. (A27) in terms of $M_{00}(3)$ and $M_{00}(5)$. By virtue of Eqs. (23), (A1) (with $p=1$), and (A11),

$$A \leqslant \frac{1}{U(2)}\left\{1+\frac{4\epsilon}{5}\left(\frac{2\pi}{3\sqrt{3}} M_{00}(3)[B(3) + B^{1/4}(3)B^{3/4}(7)]\right)^{1/2}\right\}. \tag{A29}$$

Using Eq. (A1) with $p=2q-3$, Eqs. (A8), (A11), and (A12), we obtain

$$\overline{M}_2(3) \leqslant \epsilon\left(\frac{\pi}{5\sin(\pi/5)} M_{00}(3)[B(3) + [B^{3/4}(7)B^{1/4}(11)]\right)^{1/2},$$

$$|M_{02}(5)| \leqslant \epsilon\sqrt{M_{00}(5)M_{00}(3)}[B(3)B(7)]^{1/4}. \tag{A30}$$

The functions B increase with ϵ, therefore, in Eq. (A26) we may take $\epsilon=0.02$ in order to have uniform estimates for smaller ϵ:

$$B(3)\approx 20.6,\quad B(7)\approx 52.5,\quad B(11)\approx 196.4,\quad B(13)\approx 628, \tag{A31}$$

and we are equipped for bounding $M_{00}(3)$. Inequalities (A27)–(A31) yield, finally,

$$M_{00}(3)<\frac{1.56}{(1-20\epsilon)^2}\quad \text{for}\ \epsilon\leqslant 0.02. \tag{A32}$$

Evaluation of the moment $M_{00}(3)$ in Eq. (A32) is not very accurate [the bound here is 35%, larger than $V(3)/U^2(2)$] because we exploited the interpolation (A28) for ‘‘large’’ moments M_{00}. But this only slightly affects bounds (30).

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